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Conformal Invariance for Non-Relativistic Field Theory

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Abstract

Momentum space Ward identities are derived for the amputated n-point Green's functions in $3+1$ dimensional non-relativistic conformal field theory. For $n=4$ and 6 the implications for scattering amplitudes (i.e. on-shell amputated Green's functions) are considered. Any scale invariant 2-to-2 scattering amplitude is also conformally invariant. However, conformal invariance imposes constraints on off-shell Green's functions and the three particle scattering amplitude which are not automatically satisfied if they are scale invariant. As an explicit example of a conformally invariant theory we consider non-relativistic particles in the infinite scattering length limit.

Poincaré invariant theories that are scale invariant usually have a larger symmetry group called the conformal group[1]. A similar phenomena happens for 3+1 dimensional non-relativistic systems. These are invariant under the extended Galilean group, which consists of 10 generators: translations (4), rotations (3), and Galilean boosts (3). The largest space-time symmetry group of the free Schrödinger equation is called the Schrödinger or non-relativistic conformal group [2]. This group has two additional generators corresponding to a scale transformation, and a one-dimensional special conformal transformation, sometimes called an "expansion". The infinitesimal Galilean boost, scale and conformal transformations are

$$\begin{aligned} &\text{boosts:} && \vec{x}\,' = \vec{x} + \vec{v}\,t \ , \quad t' = t \ , \\ &\text{scale:} && \vec{x}\,' = \vec{x} + s\vec{x} \ , \quad t' = t + 2st \ , \\ &\text{conformal:} && \vec{x}\,' = \vec{x} - ct\vec{x} \ , \ t' = t - ct^2 \ , \end{aligned} \tag{1}$$

where $\vec{v}$, s and c are the corresponding infinitesimal parameters. (The finite scale transformation is $\vec{x}\,' = e^s\,\vec{x}$, $t' = e^{2s}\,t$, and the finite conformal transformation is $\vec{x}\,' = \vec{x}/(1+ct)$, $1/t' = 1/t + c$.)

In this letter we explore the implications of non-relativistic conformal invariance for $3+1$ dimensional physical systems. Ward identities are derived for the amputated momentum space Green's functions. Particular attention is paid to scattering amplitudes, which are amputated on-shell momentum space Green's functions. While the off-shell Green's functions can be changed by field redefinitions, the scattering amplitudes are physical quantities and are therefore unchanged. We find that any 2-to-2 (identical particle) scattering amplitude that satisfies the scale Ward identity automatically satisfies the conformal Ward identity. However, this is not the case for the corresponding off-shell Green's function or for the 3-to-3 scattering amplitude. We construct a field theory that has a four point function which obeys the scale and conformal Ward identities. On-shell it gives S-wave scattering with an infinite scattering length.

The action for a free non-relativistic field $N(\vec{x},t)$ is

$$S_0 = \int dt d^3x \ N^\dagger \Big(i\partial_t + \frac{\nabla^2}{2M}\Big) N \, , \tag{2}$$

[1] Exceptions are known to exist, however, these theories suffer from pathologies, such as non-unitarity. A detailed discussion of scale and conformal invariance in relativistic theories can be found in Ref. [1].

where M is mass of the particle corresponding to the field N. Under an infinitesimal Galilean transformation $N'(\vec{x}',t') = (1 + iM\,\vec{v}\cdot\vec{x})N(\vec{x},t)$ or equivalently

$$\delta_g N(\vec{x},t) = N'(\vec{x},t) - N(\vec{x},t) = \mathrm{D}_g\, N(\vec{x},t) = \vec{v}\cdot\Big(iM\,\vec{x} - t\vec{\nabla}\Big)N(\vec{x},t)\,. \tag{3}$$

The action in Eq. (2) is invariant under the infinitesimal scale transformation in Eq. (1) with $N'(\vec{x}',t') = (1 - 3s/2)N(\vec{x},t)$ or equivalently

$$\delta_s N(\vec{x},t) = \mathrm{D}_s\, N(\vec{x},t) = -s\Big(\frac{3}{2} + \vec{x}\cdot\vec{\nabla} + 2t\partial_t\Big)N(\vec{x},t)\,, \tag{4}$$

and under the infinitesimal conformal transformation provided $N'(\vec{x}',t') = (1 + 3\,c\,t/2 - iM\,c\,\vec{x}^{\,2}/2)N(\vec{x},t)$ or equivalently

$$\delta_c N(\vec{x},t) = \mathrm{D}_c\, N(\vec{x},t) = c\Big(\frac{3t}{2} - \frac{iM\vec{x}^{\,2}}{2} + t\vec{x}\cdot\vec{\nabla} + t^2\partial_t\Big)N(\vec{x},t)\,. \tag{5}$$

Now consider adding interactions that preserve these invariances (an explicit example will be considered later). The position space Green's functions for the interacting theory, $G^{(2n)}(\vec{x}_i,t_i) = G^{(2n)}(\vec{x}_1,t_1;\ldots;\vec{x}_{2n},t_{2n})$, are defined by[2]

$$G^{(2n)}(\vec{x}_i,t_i) = \langle\,\Omega|\,T\Big\{\,N(\vec{x}_1,t_1)\cdots N(\vec{x}_n,t_n)N^\dagger(\vec{x}_{n+1},t_{n+1})\cdots N^\dagger(\vec{x}_{2n},t_{2n})\,\Big\}|\Omega\rangle\,, \tag{6}$$

where $|\Omega\rangle$ is the vacuum of the interacting theory and is assumed to be invariant under the Schrödinger group. Under the infinitesimal transformations in Eqs. (3-5)

$$\begin{aligned}
\delta_{(g,s,c)}G^{(2n)}(\vec{x}_i,t_i) &= \langle\Omega|\,T\Big\{\,\delta_{(g,s,c)}N(\vec{x}_1,t_1)N(\vec{x}_2,t_2)\cdots N^\dagger(\vec{x}_{2n},t_{2n})\,\Big\}|\Omega\rangle + \ldots \\
&\quad + \langle\Omega|\,T\Big\{\,N(\vec{x}_1,t_1)\cdots N^\dagger(\vec{x}_{2n-1},t_{2n-1})\delta_{(g,s,c)}N^\dagger(\vec{x}_{2n},t_{2n})\,\Big\}|\Omega\rangle \\
&= \Big[\sum_{k=1}^{n}\mathrm{D}^k_{(g,s,c)} + \sum_{k=n+1}^{2n}\mathrm{D}^{k\,\dagger}_{(g,s,c)}\Big]\langle\,\Omega|\,T\Big\{\,N(\vec{x}_1,t_1)\cdots N^\dagger(\vec{x}_{2n},t_{2n})\,\Big\}|\Omega\rangle\,,
\end{aligned} \tag{7}$$

where $\mathrm{D}^k_{(g,s,c)}$ is the differential operator for coordinates $(\vec{x}_k,t_k)$. Invariance under Galilean boosts, scale, and conformal symmetry implies that

$$\delta_{(g,s,c)}\,G^{(2n)}(\vec{x}_i,t_i) = 0\,. \tag{8}$$

The momentum space Green's functions $G^{(2n)}(\vec{p}_i,E_i) = G^{(2n)}(\vec{p}_1,E_1;\ldots;\vec{p}_{2n},E_{2n})$ are the Fourier transform of the position space Green's functions

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[2]In non-relativistic theories particle number is conserved so there must be the same number of N's as $N^\dagger$'s.

$$
G^{(2n)}(\vec{x}_1, t_1; \ldots; \vec{x}_{2n}, t_{2n}) = \Big[\prod_{k=1}^{2n} \int \frac{dE_k d^3 p_k}{(2\pi)^4} e^{-i\eta_k (E_k t_k - \vec{p}_k \cdot \vec{x}_k)} \Big]
$$

$$
\times \, (2\pi)^4 \, \delta\Big(\sum_{k=1}^{2n} \eta_k E_k \Big) \, \delta^{(3)}\Big(\sum_{k=1}^{2n} \eta_k \vec{p}_k \Big) \, G^{(2n)}(E_1, \vec{p}_1; \ldots; E_{2n}, \vec{p}_{2n}) \,, \tag{9}
$$

where η_j is 1 for incoming particles (subscripts $1, \ldots, n$) and -1 for outgoing particles (subscripts $n{+}1, \ldots, 2n$). The delta functions in Eq. (9) arise due to translational invariance. Using Eq. (8) with $\vec{x}_{2n} = 0$ and $t_{2n} = 0$ it is straightforward to show that invariance under Galilean boosts, scale transformations, and conformal transformations implies the Ward identities

$$
\mathcal{D}_{(g,s,c)} \, G^{(2n)}(E_1, \vec{p}_1; \ldots; E_{2n}, \vec{p}_{2n}) = 0 \,, \tag{10}
$$

where

$$
\mathcal{D}_g = \sum_{j=1}^{2n-1} \Big(M \vec{\nabla}_{p_j} + \vec{p}_j \frac{\partial}{\partial E_j} \Big) \,, \tag{11}
$$

$$
\mathcal{D}_s = 7n - 5 + \sum_{j=1}^{2n-1} \Big(\vec{p}_j \cdot \vec{\nabla}_{p_j} + 2E_j \, \frac{\partial}{\partial E_j} \Big) \,,
$$

$$
\mathcal{D}_c = \sum_{j=1}^{2n-1} \eta_j \, \Big(\frac{7}{2} \frac{\partial}{\partial E_j} + \frac{M}{2} \vec{\nabla}^2_{p_j} + E_j \, \frac{\partial^2}{\partial E_j{}^2} + \vec{p}_j \cdot \vec{\nabla}_{p_j} \frac{\partial}{\partial E_j} \Big) \,.
$$

In deriving Eq. (10) we have integrated over the delta functions in Eq. (9) so that

$$
E_{2n} = \sum_{j=1}^{2n-1} \eta_j E_j \,, \qquad \vec{p}_{2n} = \sum_{j=1}^{2n-1} \eta_j \vec{p}_j \,. \tag{12}
$$

The S-matrix elements are related to the amputated Green's functions $\mathcal{A}^{(2n)}(\vec{p}_i, E_i) = \mathcal{A}^{(2n)}(\vec{p}_1, E_1; \ldots; \vec{p}_{2n}, E_{2n})$ defined by[3]

$$
\mathcal{A}^{(2n)}(E_i, \vec{p}_i) = \Big[\prod_{j=1}^{2n} \Big(E_j - \frac{\vec{p}_j{}^2}{2M} \Big) \Big] \, G^{(2n)}_{con.}(E_i, \vec{p}_i) \,, \tag{13}
$$

where E_{2n} and $\vec{p}_{2n}$ are given by Eq. (12). $G^{(2n)}_{con.}$ is the connected part of $G^{(2n)}$ and also satisfies Eq. (10). Applying the Galilean boost and scale Ward identities in Eq. (10) to Eq. (13) gives

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[3]Neglecting relativistic corrections to S_0, Eq. (13) is exact because adding interactions to Eq. (2) does not effect the two point function since there is no pair creation in the non-relativistic theory.

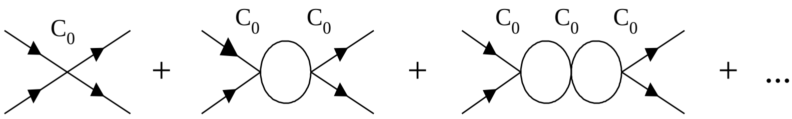

FIG. 1. Terms contributing to $\mathcal{A}^{(4)}$ from the interaction in Eq. (19).

$$\tilde{\mathcal{D}}_{(g,s)}\, \mathcal{A}^{(2n)}(E_i, \vec{p}_i) = 0\,, \tag{14}$$

where $\tilde{\mathcal{D}}_g = \mathcal{D}_g$ and

$$\tilde{\mathcal{D}}_s = 3n - 5 + \sum_{j=1}^{2n-1} \Big(\vec{p}_j \cdot \vec{\nabla}_{p_j} + 2E_j\, \frac{\partial}{\partial E_j}\Big)\,. \tag{15}$$

Applying the conformal Ward identity in Eq. (10) to Eq. (13) gives

$$\tilde{\mathcal{D}}_c \mathcal{A}^{(2n)} + \frac{1}{\sum_j \eta_j E_j - (\sum_j \eta_j \vec{p}_j)^2/(2M)} \Big[\Big(\sum_j \eta_j \vec{p}_j\Big) \cdot \mathcal{D}_g - \tilde{\mathcal{D}}_s\Big] \mathcal{A}^{(2n)} = 0\,, \tag{16}$$

where

$$\tilde{\mathcal{D}}_c = \sum_{j=1}^{2n-1} \eta_j \Big(\frac{3}{2}\, \frac{\partial}{\partial E_j} + \frac{M}{2}\, \vec{\nabla}^2_{p_j} + E_j\, \frac{\partial^2}{\partial {E_j}^2} + \vec{p}_j \cdot \vec{\nabla}_{p_j}\, \frac{\partial}{\partial E_j}\Big)\,. \tag{17}$$

Therefore, amputated Green's functions satisfying Eq. (14) also satisfy

$$\tilde{\mathcal{D}}_c\, \mathcal{A}^{(2n)} = 0\,. \tag{18}$$

The leading term in the effective field theory for non-relativistic nucleon-nucleon scattering corresponds to a scale invariant theory in the limit that the S-wave scattering lengths go to infinity (see for e.g. Ref. [3]). As we will see below, this limit corresponds to a fixed point of the renormalization group. Since in nature the S-wave scattering lengths are very large, it is the unusual scaling of operators at this non-trivial fixed point [4] that controls their importance in this effective field theory [5,6]. Motivated by this we add to Eq. (2) the interaction

$$S_1 = -\int dt d^3x\; C_0\, (N^T P N)^\dagger (N^T P N)\,, \tag{19}$$

where N is now a two component spin-1/2 fermion field and $P = i\sigma_2/2$. Higher body non-derivative interaction terms are forbidden by Fermi statistics. The interaction in Eq. (19) only mediates spin singlet S-wave NN scattering. The NN scattering amplitude arises from the sum of bubble Feynman diagrams shown in Fig. 1. The loop integration associated with a bubble has a linear ultraviolet divergence and consequently the values of the coefficients

C_0 depend on the subtraction scheme adopted. In minimal subtraction, if $p \gg 1/a$ where p is the center of mass momentum and a is the scattering length, then successive terms in the perturbative series represented by Fig. 1 get larger and larger. Subtraction schemes have been introduced where each diagram in Fig. 1 is of the same order as the sum. One such scheme is PDS [5], which subtracts not only poles at $D = 4$, but also the poles at $D = 3$ (which correspond to linear divergences). Another such scheme is the OS momentum subtraction scheme [4,7]. In these schemes the coefficients are subtraction point dependent, $C_0 \equiv C_0(\mu)$. Calculating the bubble sum in PDS or OS gives

$$\mathcal{A}^{(4)} = \frac{-C_0(\mu)}{1 + MC_0(\mu)\left[\, 2\mu - \sqrt{-4M(E_1+E_2) + (\vec{p}_1 + \vec{p}_2)^2 - i\epsilon}\, \right]/(8\pi)}\,, \tag{20}$$

where

$$C_0(\mu) = -\frac{4\pi}{M}\frac{1}{\mu - 1/a}\,. \tag{21}$$

Note that Eq. (20) holds in any frame and we have not imposed the condition that the external particles be on-shell. It is easy to see that the limit $a \to \pm\infty$ corresponds to a nontrivial ultraviolet fixed point in this scheme. If we define a rescaled coupling $\hat{C}_0 \equiv M\mu\, C_0(\mu)/(4\pi)$, then

$$\mu\frac{d}{d\mu}\hat{C}_0(\mu) = \hat{C}_0(\mu)\left[1 + \hat{C}_0(\mu)\right]. \tag{22}$$

The limit $a \to \pm\infty$ corresponds to the fixed point $\hat{C}_0 = -1$. At a fixed point one expects the theory to be scale invariant. In fact, it can be easily verified that in the $a \to \pm\infty$ limit

$$\mathcal{A}^{(4)} = \frac{8\pi}{M}\frac{1}{\sqrt{-4M(E_1+E_2) + (\vec{p}_1 + \vec{p}_2)^2 - i\epsilon}} \tag{23}$$

satisfies both the scale and conformal Ward identities in Eqs. (14) and (18). In the case of $\mathcal{A}^{(4)}$ the conformal Ward identity gives non-trivial information about the off-shell amplitude. For instance the amplitude

$$\mathcal{A}^{(4)} = \frac{8\pi}{M}\frac{1}{\sqrt{-(\vec{p}_1 - \vec{p}_3)^2 - (\vec{p}_2 - \vec{p}_3)^2 - i\epsilon}} \tag{24}$$

is scale and Galilean invariant but not conformally invariant. The expressions for $\mathcal{A}^{(4)}$ in Eqs. (23) and (24) agree on-shell, where $E_i = \vec{p}_i^{\;2}/(2M)$.

The interaction in Eq. (19) also induces non-trivial amputated Greens functions, $\mathcal{A}^{(2n)}$, for $n > 2$. (For $n = 3$ see Fig. 2.) It is believed that non-perturbatively the higher point

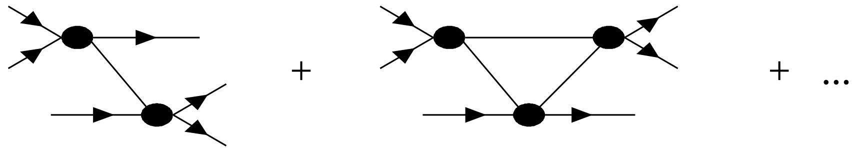

FIG. 2. Terms contributing to $\mathcal{A}^{(6)}$ from the interaction in Eq. (19). The filled circle denotes the sum of diagrams in Fig. 1.

functions are finite and we speculate that with C_0 at its critical fixed point the action S_0+S_1 defines a non-relativistic conformal field theory.

We will now derive scale and conformal Ward identities for the on-shell amplitudes since these are the physical quantities of interest. Consider the four point function. After imposing translation invariance it is a function of 12 variables

$$\mathcal{A}^{(4)}(\vec{p}_1,\vec{p}_2,\vec{p}_3,E_1,E_2,E_3)\,. \tag{25}$$

The Ward identity $\tilde{\mathcal{D}}_g\mathcal{A}^{(4)}=0$ is solved by the function $\mathcal{A}^{(4)}(\vec{p}_A,\vec{p}_B,U_1,U_2,U_3)$ where

$$\vec{p}_A=\vec{p}_1-\vec{p}_3\,,\qquad \vec{p}_B=\vec{p}_2-\vec{p}_3\,,\qquad U_i=ME_i-\frac{\vec{p}_i^{\;2}}{2}\,. \tag{26}$$

Therefore, using the Galilean boost invariance gives three constraints on $\mathcal{A}^{(4)}$ leaving 9 variables. For this function the scale and conformal identities are

$$\begin{aligned}\tilde{\mathcal{D}}_s&=1+\vec{p}_A\cdot\vec{\nabla}_{p_A}+\vec{p}_B\cdot\vec{\nabla}_{p_B}+2\sum_{j=1}^{3}U_j\frac{\partial}{\partial U_j}\,,\\ \tilde{\mathcal{D}}_c&=M\Big(-\vec{\nabla}_{p_A}\cdot\vec{\nabla}_{p_B}+\sum_{j=1}^{3}\eta_j\,U_j\frac{\partial^2}{\partial U_j^2}\Big)\,.\end{aligned} \tag{27}$$

Three more constraints are given by rotation invariance leaving a function of 6 variables, $\mathcal{A}^{(4)}(x,y,\gamma,U_1,U_2,U_3)$, where

$$x=\vec{p}_A^{\;2}\,,\qquad y=\vec{p}_B^{\;2}\,,\qquad \gamma=\vec{p}_A\cdot\vec{p}_B\,. \tag{28}$$

In terms of these variables we have

$$\begin{aligned}\tilde{\mathcal{D}}_s&=1+2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+2\gamma\frac{\partial}{\partial\gamma}+2\sum_{j=1}^{3}U_j\frac{\partial}{\partial U_j}\,,\\ \tilde{\mathcal{D}}_c&=-M\Big(\frac{\partial}{\partial\gamma}\tilde{\mathcal{D}}_s+4\gamma\frac{\partial^2}{\partial x\partial y}-\gamma\frac{\partial^2}{\partial\gamma^2}-2\sum_{j=1}^{3}U_j\frac{\partial^2}{\partial\gamma\partial U_j}-\sum_{j=1}^{3}\eta_j\,U_j\frac{\partial^2}{\partial U_j^2}\Big)\,.\end{aligned} \tag{29}$$

On-shell the four point function has an additional four constraints $U_1=U_2=U_3=0$ and $\gamma=0$, where the last condition follows because $U_4=U_1+U_2-U_3-\gamma=0$. The operator

$\tilde{\mathcal{D}}_s$ can be defined consistently on-shell since all derivatives with respect to $U_{1,2,3}$ and γ are multiplied by coefficients which vanish in the on-shell limit. In taking the on-shell limit we are assuming that derivatives of $\mathcal{A}^{(4)}$ with respect to the off-shell parameters are not singular. This is true of the explicit example in Eq. (23) as long as the momentum of the nucleons in the center of mass frame is nonzero. Finally, from Eq. (29) we see that on-shell a scale invariant $\mathcal{A}^{(4)}$ is automatically conformally invariant.

Solving $\tilde{\mathcal{D}}_s\mathcal{A}^{(4)} = 0$, the most general scattering amplitude consistent with Schrödinger group invariance is

$$A^{(4)}_{os} = \frac{1}{\sqrt{x+y}}\, F\Big(\frac{y-x}{y+x}\Big) = \frac{1}{2p}\, F(\cos\theta)\,, \tag{30}$$

where F is an arbitrary function, and θ is the scattering angle in the center of mass frame. Conformal invariance does not restrict the angular dependence of the scattering amplitude. Additional physical criteria can be used to provide further constraints. The condition that the S-wave scattering length goes to infinity corresponds to a fine tuning that produces a bound state at threshold. Assuming that this is the only fine tuning and that the interactions are short range the threshold behavior of the phase shift in the ℓth partial wave is $\delta_\ell \sim p^{2\ell+1}$ for $\ell > 0$. It is easy to see that the only partial wave obtained from Eq. (30) with acceptable threshold behavior is the S-wave, so F can be replaced by a constant. In the limit $a \to \infty$ the interaction in Eq. (19) provides an explicit example of a scale invariant theory which has this behavior.

In the case of the 3-to-3 scattering amplitude, conformal invariance will provide a new constraint independent from that of scale invariance. We proceed exactly as in the case of the 2-to-2 scattering amplitude. After imposing energy and momentum conservation the 6 point function has 20 coordinates

$$\mathcal{A}^{(6)}(\vec{p}_1, \ldots \vec{p}_5, E_1, \ldots, E_5)\,. \tag{31}$$

Using the Galilean boost invariance leaves 17 coordinates

$$\mathcal{A}^{(6)}(\vec{p}, \vec{k}, \vec{p}\,', \vec{k}\,', U_1, \ldots, U_5)\,, \tag{32}$$

where

$$\begin{aligned} \vec{p} &= \frac{2\vec{p}_3 - \vec{p}_2 - \vec{p}_1}{3}, & \vec{k} &= \vec{p}_2 - \vec{p}_1, \\ \vec{p}\,' &= \frac{2(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)}{3} - \vec{p}_4 - \vec{p}_5, & \vec{k}\,' &= \vec{p}_5 - \vec{p}_4\,. \end{aligned} \tag{33}$$

In terms of these variables

$$
\begin{aligned}
\tilde{\mathcal{D}}_s &= 4 + \vec{p}\cdot\vec{\nabla}_p + \vec{k}\cdot\vec{\nabla}_k + \vec{p}'\cdot\vec{\nabla}_{p'} + \vec{k}\,'\cdot\vec{\nabla}_{k'} + 2\sum_{j=1}^{5} U_j\,\frac{\partial}{\partial U_j} \\
\tilde{\mathcal{D}}_c &= \frac{M}{3}\Big[\vec{\nabla}_p^2 + 3\vec{\nabla}_k^2 - \vec{\nabla}_{p'}^2 - 3\vec{\nabla}_{k'}^2\Big] + M\sum_{j=1}^{5}\eta_j\,U_j\,\frac{\partial^2}{\partial U_j^2}\,.
\end{aligned}
\tag{34}
$$

After imposing rotational invariance, $\mathcal{A}^{(6)}$ should be a function of 14 variables. We have chosen

$$
\mathcal{A}^{(6)}(z_1,\ldots,z_8,\gamma,U_1,\ldots,U_5)\,,
\tag{35}
$$

where $U_i = ME_i - \vec{p}_i^{\;2}/2$ and

$$
\begin{aligned}
&z_1 = \vec{p}^{\,2}\,, &&z_2 = \vec{k}^{\,2}\,, &&z_3 = \vec{p}\,'^{\,2}\,, &&z_4 = \vec{p}\cdot\vec{k}\,, \\
&z_5 = \vec{p}\cdot\vec{p}\,'\,, &&z_6 = \vec{p}\cdot\vec{k}\,'\,, &&z_7 = \vec{k}\cdot\vec{p}\,'\,, &&z_8 = \vec{p}\,'\cdot\vec{k}\,'\,, \\
&\gamma = \vec{k}^{\,2} - \vec{k}\,'^{\,2} + 3\vec{p}^{\;2} - 3\vec{p}\,'^{\,2}\,.
\end{aligned}
\tag{36}
$$

The coordinates U_i and γ vanish on-shell since $U_6 = \sum_{j=1}^{5}\eta_j\,U_j + \gamma/4$. For the function in Eq. (35) the scale and conformal derivatives are

$$
\begin{aligned}
\tilde{\mathcal{D}}_s &= 4 + 2\sum_{j=1}^{8} z_j\,\frac{\partial}{\partial z_j} + \ldots\,, \\
\tilde{\mathcal{D}}_c &= 2M\Big(\frac{\partial}{\partial z_1} + 3\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3}\Big) + \frac{M}{3}\sum_{j,k=1}^{8} A_{jk}\frac{\partial^2}{\partial z_j\,\partial z_k} + 4M\frac{\partial}{\partial\gamma}\tilde{D}_s + \ldots\,.
\end{aligned}
\tag{37}
$$

The ellipses are terms with factors of U_i or γ and therefore vanish on-shell,

$$
A_{jk} = \begin{pmatrix}
4z_1 & 0 & 0 & 2z_4 & 2z_5 & 2z_6 & 0 & 0 \\
0 & 12z_2 & 0 & 6z_4 & 0 & 0 & 6z_7 & 0 \\
0 & 0 & -4z_3 & 0 & -2z_5 & 0 & -2z_7 & -2z_8 \\
2z_4 & 6z_4 & 0 & 3z_1+z_2 & z_7 & z_9 & 3z_5 & 0 \\
2z_5 & 0 & -2z_5 & z_7 & -z_1+z_3 & z_8 & -z_4 & -z_6 \\
2z_6 & 0 & 0 & z_9 & z_8 & z_2-3z_3 & 0 & -3z_5 \\
0 & 6z_7 & -2z_7 & 3z_5 & -z_4 & 0 & -z_2+3z_3 & -z_9 \\
0 & 0 & -2z_8 & 0 & -z_6 & -3z_5 & -z_9 & -3z_1-z_2
\end{pmatrix},
\tag{38}
$$

and $z_9 = \vec{k} \cdot \vec{k}\,'$. It is possible to express z_9 in terms of $z_1, \ldots, z_8$. For scale invariant amputated Green's functions the conformal operator can be defined on-shell because terms that involve derivatives with respect to the off-shell parameters (U_i and γ) have coefficients which vanish on-shell.

Even after demanding scale invariance the conformal Ward identity still imposes a non-trivial constraint on the amplitude. It is easy to find examples of boost and scale invariant functions which do not satisfy $\tilde{\mathcal{D}}_c \, \mathcal{A}^{(6)} = 0$. Due to the complexity of Eq. (37) we have not attempted to find its general solution.

The effective field theory for the strong interactions of nucleons is more complicated than the toy model given by $S_0 + S_1$, because nucleons have isospin degrees of freedom. With infinite spin singlet and spin triplet NN scattering lengths the four point functions are invariant under the Schrödinger group at leading order. For nucleons, the six point point functions can involve states with total spin 1/2 and 3/2. In the spin 1/2 channel a three body contact interaction with no derivatives exists and is needed to renormalize the integral equation for three body scattering [8]. This three body contact operator is expected to break scale and conformal invariance. In the spin 3/2 channel [9], no three body operator is needed and this amplitude is expected to respect the constraints from scale and conformal invariance. Explicit verification of this would be interesting.

In this letter we derived Ward identities for amputated momentum space Green's functions that follow from invariance under the Schrödinger group. We also examined implications of these constraints for 2-to-2 and 3-to-3 on-shell scattering amplitudes. Motivated by recent developments in nuclear theory, we considered a non-relativistic theory in the limit of infinite scattering length and found it gives rise to a four point function which satisfies the Ward identities which follow from Schrödinger invariance.

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